Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM (National Council of Teachers of Mathematics) process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. Make sense of problems and persevere in solving them.

Mathematically proficient students:

- start by explaining to themselves the meaning of a problem.
- look for entry points for solving problems.
- analyze givens, constraints, relationships, and goals.
- make conjectures about the form and meaning of the solution.
- plan a solution path rather than simply jumping into a solution attempt.
- consider analogous problems and try special cases and simpler forms of the original problem to gain insight into its solution.
- monitor and evaluate their progress and change course if necessary.
- transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need (middle school and high school).
- explain correspondence between equations, verbal descriptions, tables, and graphs or draw diagrams of key features and relationships, graph data, and search for regularity or trends.
- consider using concrete objects or pictures to help conceptualize and solve a problem.
- check their answers to problems using a different method, and continually ask themselves, "Does this make sense?"
- understand the approaches used by others and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students:

- make sense of quantities and their relationships in problem situations.
- decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referent.
- contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.
- create a coherent representation of the problem at hand.
- consider the units involved.
- attend to the meaning of quantities, not just how to compute them.
- flexibly use different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students:

- understand, and use stated assumptions, definitions, and previously established results in constructing arguments.
- make conjectures and build a logical progression of statements to explore the truth of their conjectures.
- analyze situations by breaking them into cases and can recognize and use counterexamples.
- justify their conclusions, communicate them to others, and respond to the arguments of others.
- reason inductively about data, making plausible arguments that take into account the context from which the data arose.
- compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is.
- construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.
- learn to determine domains to which an argument applies (high school).
- listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students:

- apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. For example, students:
    - write an addition equation to describe a situation (elementary school).
    - apply proportional reasoning to solve a problem (middle school).
    - use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another (high school).
- make assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.
- identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas.
- analyze relationships mathematically to draw conclusions.
- routinely interpret mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.

Mathematically proficient students:

- consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software.
- make sound decisions about when available tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students:
    - analyze graphs of functions and solutions generated using a graphing calculator.
    - detect possible errors by strategically using estimation and other mathematical knowledge.
    - know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data.
- identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.
- use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students:

- communicate precisely to others.
- use clear definitions in discussion with others and in their own reasoning.
- state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.
- specify units of measure, and label axes to clarify the correspondence with quantities in a problem.
- calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.
- give carefully formulated explanations to each other.
- examine claims and make explicit use of definitions.

7. Look for and make use of structure.

Mathematically proficient students:

- look closely to discern a pattern or structure. For example, students:
    - notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have (elementary school).
    - see 7×8 equals $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property (elementary school).
    - See in the expression $x^2 + 9x + 14$, that 14 is 2×7 and the 9 is $2 + 7$.
- recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems.
- step back for an overview and shift perspective.
- see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers *x* and *y*.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students:

- notice if calculations are repeated and look both for general methods and for shortcuts. For example, students:
    - notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal.
    - abstract the equation $(y - 2) / (x - 1) = 3$ by paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3 (middle school).
    - Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)\left(x^2 + x + 1\right)$, and $(x - 1)\left(x^3 + x^2 + x + 1\right)$ might lead them to the general formula for the sum of a geometric series.
- maintain oversight of the process, while attending to the details.
- evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction. The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices. In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Adapted from Common Core State Standards for Mathematics, Council of Chief State School Officers (CCSSO) and the National Governors Association (NGA)